

# Multipartite omnidirectional generalized Bell inequality

Koji Nagata

Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea

(Dated: February 5, 2008)

We derive a multipartite generalized Bell inequality which involves the entire range of settings for each of the local observers. Especially, it is applied to show non-local behavior of a six-qubit mixture of Greenberger-Horne-Zeilinger correlations stronger than previous Bell inequalities. For certain noise admixture to the correlations an explicit local realistic model exists in the case of a standard Bell experiment. Bell experiments with many local settings reveal the non-locality of the state. It turns out that the new inequality is more stringent than many other Bell inequalities in the specific quantum state.

PACS numbers: 03.65.Ud, 03.67.Mn

## I. INTRODUCTION

Non-locality in quantum physics means the possibility of distributing correlations that cannot be due to previously shared randomness, without signaling [1, 2]. Certain quantum predictions violate Bell inequalities [3], which form necessary conditions for local realistic models for the results of suitable measurements. Thus, certain measurement outcome in quantum predictions cannot admit local realistic theories.

In many cases one can build a local realistic model for the observed data. However, many such models are artificial and can be disproved if some principles of physics are taken into account. An example of such a principle is rotational invariance of correlation function — the fact that the value of correlation function does not depend on the orientation of reference frames. Taking this additional requirement into account rules out local realistic models even in situations in which standard Bell inequalities allow for an explicit construction of such models [4].

Here, we derive a generalized Bell inequality for  $N$  qubits which involves the entire range of settings for each of the local measuring apparatuses. The inequality forms a necessary condition for the existence of a local realistic model which predicts rotationally invariant correlations. Although the inequality involves the entire range of settings it can be experimentally tested using three orthogonal local measurement settings. This is a direct consequence of the assumed form of rotationally invariant correlations.

Next, we consider a mixture of Greenberger-Horne-Zeilinger (GHZ) states [5] written in three orthogonal directions. A white noise is added to the mixture with some probability. We take the minimal amount of noise admixture for which one does not violate a Bell inequality as a measure of the strength of the inequality. It turns out that the new inequality is more stringent than many other inequalities [4, 6, 7, 8] in the specific quantum state.

## II. MULTIPARTITE OMNIDIRECTIONAL GENERALIZED BELL INEQUALITY

Consider  $N$  spin- $\frac{1}{2}$  particles, each in a separate laboratory. Let us parameterize the local settings of the  $j$ th observer with a unit vector  $\vec{n}_j$  with  $j = 1, \dots, N$ . One can introduce the “Bell” correlation function, which is the average of the product of the local results

$$E(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_N) = \langle r_1(\vec{n}_1) r_2(\vec{n}_2) \cdots r_N(\vec{n}_N) \rangle_{\text{avg}}, \quad (1)$$

where  $r_j(\vec{n}_j)$  is the local result,  $\pm 1$ , which is obtained if the measurement direction is set at  $\vec{n}_j$ . If correlation function admits a rotationally invariant tensor structure familiar from quantum mechanics, we can introduce the following form:

$$E(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_N) = \hat{T} \cdot (\vec{n}_1 \otimes \vec{n}_2 \otimes \cdots \otimes \vec{n}_N), \quad (2)$$

where  $\otimes$  denotes the tensor product,  $\cdot$  the scalar product in  $\mathbb{R}^{3N}$  and  $\hat{T}$  is the correlation tensor the elements of which are given by

$$T_{i_1 \dots i_N} \equiv E(\vec{x}_1^{(i_1)}, \vec{x}_2^{(i_2)}, \dots, \vec{x}_N^{(i_N)}), \quad (3)$$

where  $\vec{x}_j^{(i_j)}$  is a unit vector of the local coordinate system of the  $j$ th observer;  $i_j = 1, 2, 3$  gives the full set of orthogonal vectors defining the local Cartesian coordinates. The components of the correlation tensor are experimentally accessible by measuring the correlation function at the directions given by the bases vectors in which the tensor is written [11]. Suppose one knows the values of all  $3^N$  components of the correlation tensor,  $T_{i_1 \dots i_N}$ . Then, with the help of formula (2) one can compute the value of the correlation function for all other possible sets of local settings.

We shall derive a necessary condition for the existence of a local realistic description of the rotationally invariant correlation function (2). A correlation function has a local realistic model if it can be written as

$$E_{LR}(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_N) = \int d\lambda \rho(\lambda) I^{(1)}(\vec{n}_1, \lambda) I^{(2)}(\vec{n}_2, \lambda) \cdots I^{(N)}(\vec{n}_N, \lambda), \quad (4)$$

where  $\lambda$  denotes a set of hidden variables,  $\rho(\lambda)$  is their distribution, and  $I^{(j)}(\vec{n}_j, \lambda)$  is the predetermined ‘‘hidden’’ result of the measurement of all the dichotomic observables parameterized by any direction of  $\vec{n}_j$ .

One can write the observable (unit) vector  $\vec{n}_j$  in a spherical coordinate system:

$$\vec{n}_j(\theta_j, \phi_j) = \sin \theta_j \cos \phi_j \vec{x}_j^{(1)} + \sin \theta_j \sin \phi_j \vec{x}_j^{(2)} + \cos \theta_j \vec{x}_j^{(3)}, \quad (5)$$

where  $\vec{x}_j^{(1)}$ ,  $\vec{x}_j^{(2)}$ , and  $\vec{x}_j^{(3)}$  are the Cartesian axes relative to which spherical angles are measured.

We shall show that the scalar product of the local realistic correlation function,  $E_{LR}$  given in (4), with the rotationally invariant correlation function,  $E$  given in (2), is bounded by a specific number dependent on  $\hat{T}$ . We use decomposition (5) and introduce the usual measure  $d\Omega_j = \sin \theta_j d\theta_j d\phi_j$  for the system of the  $j$ th observer. It will be proven that

$$(E_{LR}, E) = \int d\Omega_1 \cdots \int d\Omega_N E_{LR}(\theta_1, \phi_1, \dots, \theta_N, \phi_N) \times E(\theta_1, \phi_1, \dots, \theta_N, \phi_N) \leq (2\pi)^N T_{\max}, \quad (6)$$

where  $T_{\max}$  is the maximal possible value of the correlation tensor component, maximized over choices of all possible local settings:

$$T_{\max} = \max_{\theta_1, \phi_1, \dots, \theta_N, \phi_N} E(\theta_1, \phi_1, \dots, \theta_N, \phi_N). \quad (7)$$

A necessary condition for the existence of a local realistic description of the rotationally invariant correlation function, i.e., for  $E_{LR}$  to be equal to  $E$ , is that the following scalar products are equal:  $(E_{LR}, E) = (E, E)$ . If one finds  $(E_{LR}, E) < (E, E)$ , then the rotationally invariant correlation function cannot be explained by any local realistic theory. Note that, due to the integrations in (6), we are looking for a model for the entire range of settings.

In what follows, we derive the upper bound of (6). Since the local realistic model is an average over  $\lambda$ , it is enough to find the upper bound of the following expression:

$$\int d\Omega_1 \cdots \int d\Omega_N I^{(1)}(\theta_1, \phi_1) \cdots I^{(N)}(\theta_N, \phi_N) \times \sum_{i_1, i_2, \dots, i_N=1,2,3} T_{i_1 i_2 \dots i_N} c_1^{i_1} c_2^{i_2} \cdots c_N^{i_N}, \quad (8)$$

where

$$\vec{c}_j = (c_j^1, c_j^2, c_j^3) = (\sin \theta_j \cos \phi_j, \sin \theta_j \sin \phi_j, \cos \theta_j), \quad (9)$$

and

$$T_{i_1 i_2 \dots i_N} = \hat{T} \cdot (\vec{x}_1^{(i_1)} \otimes \vec{x}_2^{(i_2)} \otimes \cdots \otimes \vec{x}_N^{(i_N)}), \quad (10)$$

compare with Eq. (2) and Eq. (3). Here, we use the abbreviation  $I^{(j)}(\theta_j, \phi_j)$  for  $I^{(j)}(\vec{n}_j(\theta_j, \phi_j), \lambda)$ .

Let us analyze the structure of expression (8). Notice that (8) is a sum, with coefficients given by  $T_{i_1 i_2 \dots i_N}$ , of products of the following integrals:  $\int d\Omega_j I^{(j)}(\theta_j, \phi_j) \sin \theta_j \cos \phi_j$ ,  $\int d\Omega_j I^{(j)}(\theta_j, \phi_j) \sin \theta_j \sin \phi_j$ , and  $\int d\Omega_j I^{(j)}(\theta_j, \phi_j) \cos \theta_j$ . These integrals are scalar products of  $I^{(j)}(\theta_j, \phi_j)$  with three orthogonal functions. One has  $\int d\Omega_j c_j^{i_k} c_j^{i'_k} = (4\pi/3)\delta_{i_k, i'_k}$ . The normalized functions  $\sqrt{3/4\pi} \sin \theta_j \cos \phi_j$ ,  $\sqrt{3/4\pi} \sin \theta_j \sin \phi_j$ , and  $\sqrt{3/4\pi} \cos \theta_j$  form a basis of a three-dimensional real functional space, which we shall call  $S^{(3)}$  [9]. Using these three functions one can write the projection of function  $I^{(j)}(\theta_j, \phi_j)$  onto them as

$$\begin{aligned} \int d\Omega_j I^{(j)}(\theta_j, \phi_j) \sqrt{3/4\pi} \sin \theta_j \cos \phi_j &= \sin \beta_j \cos \gamma_j \|I^{(j)}\|, \\ \int d\Omega_j I^{(j)}(\theta_j, \phi_j) \sqrt{3/4\pi} \sin \theta_j \sin \phi_j &= \sin \beta_j \sin \gamma_j \|I^{(j)}\|, \\ \int d\Omega_j I^{(j)}(\theta_j, \phi_j) \sqrt{3/4\pi} \cos \theta_j &= \cos \beta_j \|I^{(j)}\|, \end{aligned} \quad (11)$$

where  $\|I^{(j)}\|$  is the length of the projection, and  $\beta_j$  and  $\gamma_j$  are some angles. Going back to expression (8) one has

$$\begin{aligned} &\left(\frac{4\pi}{3}\right)^{N/2} \prod_{j=1}^N \|I^{(j)}\| \\ &\times \sum_{i_1, i_2, \dots, i_N=1,2,3} T_{i_1 i_2 \dots i_N} e_1^{i_1} e_2^{i_2} \cdots e_N^{i_N}, \end{aligned} \quad (12)$$

with a normalized vector

$$(e_j^1, e_j^2, e_j^3) = (\sin \beta_j \cos \gamma_j, \sin \beta_j \sin \gamma_j, \cos \beta_j). \quad (13)$$

Note that the sum in (12) over the components of this vector is just  $\hat{T} \cdot (\vec{e}_1 \otimes \vec{e}_2 \otimes \cdots \otimes \vec{e}_N)$ , i.e., it is a component of the tensor  $\hat{T}$  in the local Cartesian coordinate systems specified by the vectors  $\vec{e}_j$ . If one knows all the values of  $T_{i_1 i_2 \dots i_N}$ , one can always find the maximal possible value of such a component, and it is equal to  $T_{\max}$ , of Eq. (7). Thus,

$$\sum_{i_1, i_2, \dots, i_N=1,2,3} T_{i_1 i_2 \dots i_N} e_1^{i_1} e_2^{i_2} \cdots e_N^{i_N} \leq T_{\max}. \quad (14)$$

It remains to show the upper bound on the norm  $\|I^{(j)}\|$ . From the definition the norm is given by a maximal possible value of the scalar product between  $I^{(j)}(\theta_j, \phi_j)$  and any normalized function belonging to  $S^{(3)}$ :

$$\|I^{(j)}\| = \max_{|\vec{d}|=1} \left[ \sqrt{\frac{3}{4\pi}} \int d\Omega_j I^{(j)}(\theta_j, \phi_j) \sum_{k=1}^3 d_k c_j^k \right], \quad (15)$$

where  $\vec{d} = (d_1, d_2, d_3)$  and  $|\vec{d}| = \sqrt{d_1^2 + d_2^2 + d_3^2} = 1$ . Since  $|I^{(j)}(\theta_j, \phi_j)| = 1$ , one has for the integral of the modulus

$$\|I^{(j)}\| \leq \max_{|\vec{d}|=1} \left[ \sqrt{\frac{3}{4\pi}} \int d\Omega_j |\vec{d} \cdot \vec{c}_j| \right], \quad (16)$$

where the dot between three-dimensional vectors denotes the usual scalar product in  $\mathbb{R}^3$ . The values of this scalar product are then integrated (summed) over all values of  $\theta_j$  and  $\phi_j$ , i.e., over vectors  $\vec{c}_j$  on the whole sphere. Since the measure is rotationally invariant the integral does not depend on particular  $\vec{d}$  and we choose it as a unit vector in direction  $\vec{z}$ . For this choice

$$\|I^{(j)}\| \leq \int d\Omega_j \left| \sqrt{\frac{3}{4\pi}} \cos \theta_j \right| = 2\pi \sqrt{\frac{3}{4\pi}}. \quad (17)$$

Finally  $(E_{LR}, E) \leq (2\pi)^N T_{\max}$ .  $\square$

The relation (6) is a generalized  $N$ -qubit Bell inequality with the entire range of measurement settings. Specific local hidden variable models,  $E_{LR}$ , which rebuild rotationally invariant correlations,  $E$ , satisfy it. However, there exist rotationally invariant correlations which cannot be modeled in a local realistic way. Whenever the scalar product  $(E, E)$  is bigger than the product  $(E_{LR}, E)$  there can be no local realistic model for  $E$ . Thus, we compute

$$\begin{aligned} (E, E) &= \int d\Omega_1 \cdots \int d\Omega_N \left( \sum_{i_1, \dots, i_N=1}^3 T_{i_1 \dots i_N} c_1^{i_1} \cdots c_N^{i_N} \right)^2 \\ &= (4\pi/3)^N \sum_{i_1, \dots, i_N=1}^3 T_{i_1 \dots i_N}^2, \end{aligned} \quad (18)$$

where we have used the orthogonality relation  $\int d\Omega_j c_j^{i_k} c_j^{i'_k} = (4\pi/3)\delta_{i_k, i'_k}$ . Finally, the necessary condition for the existence of a local realistic model of rotationally invariant correlations which involve the entire range of settings reads

$$\max \sum_{i_1, i_2, \dots, i_N=1, 2, 3} T_{i_1 i_2 \dots i_N}^2 \leq \left(\frac{3}{2}\right)^N T_{\max}, \quad (19)$$

where the maximization is taken over all independent rotations of local coordinate systems (or equivalently over all possible measurement directions).

### III. MIXTURE OF SIX-QUBIT GHZ STATE

Now, we shall present the specific quantum state for which the newly derived inequality is better than the previous inequalities described in Refs. [4, 6, 7, 8].

Consider the following six-qubit GHZ state

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} (|z+\rangle_1 \dots |z+\rangle_5 |z-\rangle_6 + |z-\rangle_1 \dots |z-\rangle_5 |z+\rangle_6), \quad (20)$$

where  $|z\pm\rangle_j$  is the eigenstate of the local  $\sigma_z$  operator of the  $j$ th observer. Note that the states of the last party are flipped with respect to the states of the other parties. We rotate the states of all individual qubits by the angle  $\alpha = 2\pi/3$  around the axis  $\vec{m} = \frac{1}{\sqrt{3}}(1, 1, 1)$  on the Bloch

sphere. This rotation cyclically permutes the directions of the Cartesian coordinate system. The unitary realizing this rotation is given by [10]:

$$U = e^{-i\frac{\alpha}{2}\vec{m}\cdot\vec{\sigma}} = \frac{1}{2} \begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}, \quad (21)$$

with  $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  being a vector of local Pauli operators. Applying  $U$  to all the qubits gives a new state  $|\psi_1\rangle \equiv U^{\otimes 6}|\psi_3\rangle$ . With the double application one gets  $|\psi_2\rangle \equiv U^{\otimes 6}|\psi_1\rangle$ . The states  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are, up to a global phase which does not contribute to correlations, of the same form as  $|\psi_3\rangle$ , but are written in the local bases of  $\sigma_x$  and  $\sigma_y$  operators, respectively. Finally, one introduces a mixture of Greenberger-Horne-Zeilinger correlations and white noise:

$$\rho = \frac{f}{3} \sum_{k=1}^3 |\psi_k\rangle\langle\psi_k| + (1-f)\rho_{\text{noise}}. \quad (22)$$

We are interested in six-qubit correlations of this state. The correlation tensor has  $3 \cdot \left(\binom{6}{2} + \binom{6}{4}\right) + 3 = 93$  nonvanishing six-qubit components. These are  $\binom{6}{2} + \binom{6}{4}$  components with two equal indices different than the remaining four equal indices, e.g.,  $T_{111122}, T_{121121}, T_{112222}, \dots$  There are three such sets which correspond to the three possible different pairs of indices, i.e.,  $\{1, 2\}, \{1, 3\}$  and  $\{2, 3\}$  (e.g.,  $T_{111122}, T_{113311}, T_{223333}, \dots$ ). In the remaining three components all indices are the same. The value of every component is given by  $\pm f/3$ . Thus, the maximal possible component of the correlation tensor is equal to  $T_{\max} = f/3$ .

For certain noise admixture the mixed state (22) admits a local realistic model for correlations obtained in a Bell experiment with any two local settings. The sufficient condition for the existence of such a model is that the components of the correlation tensor, maximized over the choice of all local coordinate systems, satisfy [8]:

$$\max \sum_{i_1, \dots, i_6=1}^2 T_{i_1 \dots i_6}^2 \leq 1. \quad (23)$$

The state has 32 components which contribute to this sum. Thus, the left-hand side equals  $\frac{32}{9}f^2$  and the condition is satisfied for  $f \leq \frac{3}{4\sqrt{2}} = 0.53033$ .

However, one can still observe non-local behavior of the state if measurements of more local settings are allowed even though one adds more noise to the state. First consider a Bell experiment in which all settings from arbitrary chosen local planes are measured. A similar technique to the one described here (with less general integrations) leads to the necessary condition for local realistic models which are rotationally invariant with respect to the measured correlations [4]:

$$\max \sum_{i_1, \dots, i_6=1}^2 T_{i_1 \dots i_6}^2 \leq \left(\frac{4}{\pi}\right)^6 T_{\max}, \quad (24)$$

where now the maximization is taken over all possible positions of local measurement planes, and  $T_{\max}$  is computed in the plane for which the left-hand side is maximal. For the state under consideration the left-hand side of this condition is the same as the left-hand side of (23), and one directly finds that the necessary condition (24) is violated for  $f > 0.399422$ . For lower values of  $f$  it could be that the specific local realistic model, proven to exist before, can be extended to measurements within the plane.

Nevertheless, the new inequality increases the range of  $f$  for which the extension is impossible. This is due to the fact that the settings over the whole Bloch sphere are allowed. For the considered state the left-hand side of condition (19) is the sum of 93 terms, and thus equals  $\frac{93}{9}f^2$ , which gives violation of this condition whenever  $f > 0.36744$ .

#### IV. SUMMARY

In summary, we derived a generalized  $N$ -particle Bell inequality which involves the entire range of settings for each of the local measuring apparatuses. The new inequality better reveals the impossibility of a local realistic model for correlations in a specific quantum state, i.e., a mixture of Greenberger-Horne-Zeilinger states than

many previous inequalities. We illustrate this by the six-qubit state. We illustrate this by the six-qubit state. In this case, for a certain noise admixture, one can explicitly build a local realistic model for the correlations obtained in a standard Bell experiment – the experiment with two local settings – independently of the plane which is spanned by the settings. The inequalities which take into account the entire range of settings in local planes disprove the possibility of the model for a substantially bigger range of noise admixture. This range can be further enlarged using inequalities which involve correlations between observables from the whole Bloch sphere.

It is very interesting to consider the following. Could there be more examples such that this Bell inequality is more stringent? Could this Bell inequality distinguish between different classes of multipartite quantum states? What about degree of entanglement for these specific quantum states?

#### Acknowledgments

The author thanks M. Żukowski, W. Laskowski, M. Wieśniak, and T. Paterek for helpfull discussions. This work has been supported by Frontier Basic Research Programs at KAIST and K.N. is supported by the BK21 research professorship.

---

[1] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, The Netherlands, 1993).  
[2] M. Redhead, *Incompleteness, Nonlocality, and Realism* (Clarendon Press, Oxford, 1989), 2nd ed.  
[3] J. S. Bell, Physics **1**, 195 (1964).  
[4] K. Nagata, W. Laskowski, M. Wieśniak, and M. Żukowski, Phys. Rev. Lett. **93**, 230403 (2004).  
[5] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989), pp. 69-72.  
[6] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).  
[7] H. Weinfurter and M. Żukowski, Phys. Rev. A **64**, 010102(R) (2001).  
[8] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).  
[9] D. Kaszlikowski and M. Żukowski, Phys. Rev. A **61**, 22114 (2000).  
[10] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).  
[11] The same idea is behind quantum tomography.